Problem Set 1 Solutions

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Problem 1: Quantum gate warm-up

a) Translating the circuit to unitary operators:

$$\begin{aligned}
CNOT(H\otimes I)|00\rangle &= CNOT|+0\rangle \\
&= CNOT\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)|0\rangle \\
&= \frac{1}{\sqrt{2}}(CNOT|00\rangle + CNOT|10\rangle) \\
&= \frac{1}{\sqrt{2}}(|00\rangle+|11\rangle) \\
&= |\Phi_+\rangle
\end{aligned}$$

Similarly, $|01\rangle \mapsto |\Psi^+\rangle$, $|10\rangle \mapsto |\Phi_-\rangle$, $|11\rangle \mapsto |\Psi_-\rangle$.

b) There are a couple of ways to do this. You can demonstrate equality by showing that the circuits act on the computational basis equivalently. For example, for the first circuit:

$$\begin{aligned}
(H\otimes H)CNOT(H\otimes H)|x\rangle|y\rangle &= (H\otimes H)CNOT\frac{1}{2}(|0\rangle+(-1)^x|1\rangle)(|0\rangle+(-1)^y|1\rangle) \\
&= \frac{1}{2}(H\otimes H)CNOT(|00\rangle+(-1)^x|10\rangle+(-1)^y|01\rangle+(-1)^{x+y}|11\rangle) \\
&= \frac{1}{2}(H\otimes H)(|00\rangle+(-1)^x|11\rangle+(-1)^y|01\rangle+(-1)^{x+y}|10\rangle) \\
&= \frac{1}{2}(|++\rangle+(-1)^x|--\rangle+(-1)^y|+-\rangle+(-1)^{x+y}|-+\rangle) \\
&= \frac{1}{2}(|+\rangle\left(|+\rangle+(-1)^y|-\rangle\right)+|-\rangle\left((-1)^{x+y}|+\rangle+(-1)^x|-\rangle\right)) \\
&= \frac{1}{2}(|+\rangle\left(|+\rangle+(-1)^y|-\rangle\right)+(-1)^{x+y}|-\rangle\left(|+\rangle+(-1)^y|-\rangle\right)) \\
&= \frac{1}{2}(|+\rangle+(-1)^{x+y}|-\rangle)\otimes(|+\rangle+(-1)^y|-\rangle) \\
&= \frac{1}{4}\left((1+(-1)^{x+y})|0\rangle+(1-(-1)^{x+y})|1\rangle\right) \\
&\qquad \otimes\left((1+(-1)^y)|0\rangle+(1-(-1)^y)|1\rangle\right) \\
&= |x\oplus y\rangle|y\rangle
\end{aligned}$$

This is equal to CNOT where the second qubit is the control qubit. I kept the bits x and y arbitrary, but you could also demonstrate equality by letting $x, y \in \{0,1\}$ for each possible value.

You could also use matrix representations. For example, for the second circuit:

$$\begin{aligned}
CNOT(Z\otimes I)CNOT &\equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
&\equiv Z\otimes I
\end{aligned}$$

The other circuits are similar.

Problem 2: Grover search

a) The best you can do is keep choosing entries in any order until you find the entry ω. The probability of finding ω is greater than $1/2$ if you check at least $N/2$ entries (and therefore query f_ω at least $N/2$ times).

c) $\langle\omega|s\rangle = 1/\sqrt{N}$

d) Geometrically, first U_ω reflects $|s\rangle$ about $|\omega^\perp\rangle$, and then U_s reflects this vector about $|s\rangle$. The end result is that $|s\rangle$ rotates by an angle 2θ toward $|\omega\rangle$.

e) R^k causes $|s\rangle$ to rotate by an angle $2\theta k$ counterclockwise.

f) We want $2\theta k + \theta = \pi/2$. Using $\theta \approx \sin\theta = 1/\sqrt{N}$, we thus want

$$(2k+1)\frac{1}{\sqrt{N}} = \frac{\pi}{2} \qquad \Rightarrow \qquad k = \frac{1}{2}\left(\frac{\pi}{2}\sqrt{N} - 1\right) \approx \frac{\pi}{4}\sqrt{N}$$

I.e. we should choose k to be the closest integer to the answer above. Notice that k scales as $\sqrt{N}$, instead of N as we found in part a). Therefore, Grover Search gives us a modest speed-up over the optimal classical algorithm.

Extra Problem: Distance measures

a) Let $\sigma = \rho - \tilde{\rho}$ and $\sigma|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$ be an orthonormal basis of eigenvectors of σ. By definition,

$$d(\rho,\tilde{\rho}) = \frac{1}{2}\sum_{a=1}^{N}|p_a - \tilde{p}_a| = \frac{1}{2}\sum_{a=1}^{N}|\operatorname{Tr}(\sigma E_a)|.$$

Now

$$|\operatorname{Tr}(\sigma E_a)| = \left|\sum_{i=1}^{N}\langle\lambda_i|\sigma E_a|\lambda_i\rangle\right| = \left|\sum_{i=1}^{N}\lambda_i\langle\lambda_i|E_a|\lambda_i\rangle\right| \le \sum_{i=1}^{N}|\lambda_i|\langle\lambda_i|E_a|\lambda_i\rangle,$$

so

$$\sum_{a=1}^{N}|\operatorname{Tr}(\sigma E_a)| \le \sum_{a=1}^{N}\sum_{i=1}^{N}|\lambda_i|\langle\lambda_i|E_a|\lambda_i\rangle = \sum_{i=1}^{N}|\lambda_i|\langle\lambda_i|\left(\sum_{a=1}^{N}E_a\right)|\lambda_i\rangle = \sum_{i=1}^{N}|\lambda_i|.$$

We therefore obtain $d(p, \tilde{p}) \leq \frac{1}{2}\sum_{i=1}^{N}|\lambda_i|$.

b) Choosing $E_a = |\lambda_a\rangle\langle\lambda_a|$ saturates the bound.

c)

$$\begin{aligned}\|\rho - \tilde{\rho}\|_1 &= \mathrm{Tr}\left[(\sigma^\dagger\sigma)^{1/2}\right] \\ &= \mathrm{Tr}\,\mathrm{diag}(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_N|) \qquad \text{in the } \{|\lambda_i\rangle\} \text{ basis} \\ &= \sum_{i=1}^{N}|\lambda_i|\end{aligned}$$

So, $d(\rho, \tilde{\rho}) = \frac{1}{2}\|\rho - \tilde{\rho}\|_1$.

d)

$$\rho = \begin{pmatrix} \cos^2(\theta/2) & \cos(\theta/2)\sin(\theta/2) \\ \cos(\theta/2)\sin(\theta/2) & \sin^2(\theta/2) \end{pmatrix}$$

$$\tilde{\rho} = \begin{pmatrix} \sin^2(\theta/2) & \cos(\theta/2)\sin(\theta/2) \\ \cos(\theta/2)\sin(\theta/2) & \cos^2(\theta/2) \end{pmatrix}$$

Therefore, $d(\rho, \tilde{\rho}) = |\cos^2(\theta/2) - \sin^2(\theta/2)|$.

e)

$$\begin{aligned}\||\psi\rangle - |\tilde{\psi}\rangle\|_2^2 &= (\cos(\theta/2) - \sin(\theta/2))^2 + (\sin(\theta/2) - \cos(\theta/2))^2 \\ &= 2(\cos(\theta/2) - \sin(\theta/2))^2\end{aligned}$$

So, $\||\psi\rangle - |\tilde{\psi}\rangle\|_2 = \sqrt{2}|\cos(\theta/2) - \sin(\theta/2)|$.

$$\begin{aligned}d(\rho, \tilde{\rho}) &= |\cos(\theta/2) - \sin(\theta/2)| \cdot |\cos(\theta/2) + \sin(\theta/2)| \\ &\leq |\cos(\theta/2) - \sin(\theta/2)| \cdot \sqrt{2} \\ &= \||\psi\rangle - |\tilde{\psi}\rangle\|_2\end{aligned}$$

f) E.g. plug in $\theta = 3\pi/2$, then $|\psi\rangle = -|\tilde{\psi}\rangle$. In other words, the two states differ only by a phase, so they are the same state, physically. One finds that $d(|\psi\rangle, |\tilde{\psi}\rangle) = 0$, but $\||\psi\rangle - |\tilde{\psi}\rangle\|_2 = 2$. The 2-norm fails to distinguish such states.