Quantum Information – Problem Set 2

Advanced Quantum Mechanics – KU Leuven
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Problem 1: The Schmidt decomposition

Let $\mathcal{H}_{AB}$ be a separable Hilbert space, i.e. it admits a countable basis of orthonormal eigenvectors. Furthermore, suppose that $\mathcal{H}_{AB}$ factorizes into the tensor product $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, and let $|\psi\rangle_{AB} \in \mathcal{H}_{AB}$. We can always write

$$|\psi\rangle_{AB} = \sum_i \sum_\mu a_{i\mu} |i\rangle_A |\mu\rangle_B \tag{1}$$

where $\{|i\rangle_A\}$ and $\{|\mu\rangle_B\}$ are orthonormal bases for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. For each i, let us define the vector $|\tilde{i}\rangle_B = \sum_\mu a_{i\mu}|\mu\rangle_B$, so that

$$|\psi\rangle_{AB} = \sum_i |i\rangle_A |\tilde{i}\rangle_B. \tag{2}$$

Note that the $|\tilde{i}\rangle_B$ need not be normalized nor orthogonal.

a) Suppose that $\{|i\rangle_A\}$ is the basis in which $\rho_A = \mathrm{Tr}_B\, |\psi\rangle_{AB}\langle\psi|_{AB}$ is diagonal, and let the set S label the non-zero eigenvalues of ρ_A, i.e. $p_i \neq 0 \Leftrightarrow i \in S$. In other words,

$$\rho_A = \sum_{i\in S} p_i |i\rangle_A\langle i|_A. \tag{3}$$

Starting from Eq. (2), compute ρ_A by taking the partial trace over B and show that

$$\rho_A = \sum_i \sum_{i'} \langle \tilde{i}' | \tilde{i} \rangle_B |i\rangle_A \langle i'|_A. \tag{4}$$

b) Compare Eqs. (3) and (4). What do you conclude about the overlap $\langle \tilde{i}' | \tilde{i} \rangle$? Use this to write down a set of orthonormal vectors in B.

c) Write down $|\psi\rangle_{AB}$ using the basis $\{|i\rangle\}_A$ and the orthonormal set of vectors in B that you found above. What are the eigenvalues of ρ_B?

Note: This important result is known as the *Schmidt decomposition*. Any bipartite pure state $|\psi\rangle_{AB}$ can be written in the form

$$|\psi\rangle_{AB} = \sum_j \sqrt{p_j} |\phi_j\rangle_A |\chi_j\rangle_B, \tag{5}$$

where the vectors $|\phi_j\rangle_A$ and $|\chi_j\rangle_B$ are orthonormal in A and B, separately. Note that this decomposition is state-dependent. In general, if $|\omega\rangle_{AB}$ is some other state, then it will not have such a decomposition in terms of the same vectors.

Problem 2: Basic properties of entanglement entropy

Von Neumann entropy is an incredibly important concept in quantum information science, and any overview of the field would be incomplete without at least touching on it. In this set of problems, we'll learn about what it is, about some of its properties, and how it's useful.

Definition 1 *Let $\mathcal{H}$ be a separable Hilbert space and let $\rho \in \mathcal{L}(\mathcal{H})$ be a density matrix, i.e. ρ is Hermitian, positive semidefinite, and satisfies* $\operatorname{Tr} \rho = 1$*. The* Von Neumann entropy *of ρ is*

$$S(\rho) = -\operatorname{Tr} \rho \log \rho. \tag{6}$$

If $\{|p_i\rangle\}$ is an orthonormal basis of eigenstates of ρ, so that $\rho|p_i\rangle = p_i|p_i\rangle$, and hence $\rho = \sum_i p_i |p_i\rangle\langle p_i|$ (where some of the p_i may be zero), then

$$S(\rho) = -\sum_i p_i \log p_i. \tag{7}$$

In particular, if $\mathcal{H}$ is bipartite, i.e. $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, so that we can define the reduced density matrices $\rho_A = \operatorname{Tr}_B \rho$ and $\rho_B = \operatorname{Tr}_A \rho$, then $S(\rho_A)$ and $S(\rho_B)$ are the *entanglement entropies* of ρ_A and ρ_B, respectively.

The reason for this terminology is that entanglement entropy is generally a measure of entanglement. It turns out that for a bipartite Hilbert space, if the total state ρ is pure, then entanglement entropy is essentially the *unique* quantitative measure of how much A and B are entangled. Let's investigate this claim a bit in this problem.

To simplify the problem, let $\mathcal{H}$ be a finite-dimensional Hilbert space with $\dim \mathcal{H} = N$.

a) What is the maximum value of $S(\rho)$? Which density matrix ρ achieves this value?

b) Show that $S(\rho) = 0$ if and only if ρ is a pure state, i.e. $\rho = |\psi\rangle\langle\psi|$ for some vector $|\psi\rangle$.

c) Now suppose that $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. If ρ is a pure state, show that $S(\rho_A) = S(\rho_B)$, where $\rho_A = \operatorname{Tr}_B \rho$ and $\rho_B = \operatorname{Tr}_A \rho$. (*Hint:* use the result of Problem 1.)

d) Using the results of parts (b) and (c), show that if ρ is a pure state, then A and B are unentangled if and only if $S(\rho_A) = S(\rho_B) = 0$.

e) Show that $S(U\rho U^\dagger) = S(\rho)$ for any unitary operator U. This shows that Von Neumann entropy is invariant under local operations.

Problem 3: Positivity of relative entropy and subadditivity

a) Show that $\log x \le x - 1$ for all positive real numbers, with equality if and only if $x = 1$.

b) The classical relative entropy of a probability distribution $\{p(x)\}$ relative to $\{q(x)\}$ is defined as

$$H(p \,\|\, q) = \sum_x p(x) \left(\log p(x) - \log q(x)\right). \tag{8}$$

Show that

$$H(p \,\|\, q) \ge 0, \tag{9}$$

with equality if and only if the distributions are identical. (*Hint:* apply the inequality from (a) to $\log(q(x)/p(x))$.)

c) The quantum relative entropy of the density operator ρ with respect to σ is

$$S(\rho \,\|\, \sigma) = \mathrm{Tr}\left[\rho(\log \rho - \log \sigma)\right]. \tag{10}$$

Let $\{p_i\}$ denote the eigenvalues of ρ and $\{q_a\}$ denote the eigenvalues of σ. Show that

$$S(\rho \,\|\sigma) = \sum_i p_i \left(\log p_i - \sum_a D_{ia} \log q_a\right), \tag{11}$$

where D_{ia} is a doubly stochastic matrix. Express D_{ia} in terms of the eigenstates of ρ and σ. (A matrix is doubly stochastic if its entries are nonnegative real numbers, where each row and each column sums to one.)

d) Show that if D_{ia} is doubly stochastic, then (for each i)

$$\log\left(\sum_a D_{ia} q_a\right) \ge \sum_a D_{ia} \log q_a, \tag{12}$$

with equality only if $D_{ia} = 1$ for some a.

e) Show that

$$S(\rho \,\|\, \sigma) \ge H(p \,\|r), \tag{13}$$

where $r_i = \sum_a D_{ia} q_a$.

f) Show that $S(\rho \,\|\, \sigma) \ge 0$, with equality if and only if $\rho = \sigma$.

g) Use nonnegativity of quantum relative entropy to prove the *subadditivity* of Von Neumann entropy,

$$S(\rho_{AB}) \le S(\rho_A) + S(\rho_B), \tag{14}$$

with equality if and only if $\rho_{AB} = \rho_A \otimes \rho_B$. (*Hint:* Consider the relative entropy of ρ_{AB} and $\rho_A \otimes \rho_B$.)