Quantum Information – Problem Set 1

Advanced Quantum Mechanics – KU Leuven
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Problem 1: Fun with entanglement

One feature that distinguishes quantum computation from classical computation is that a quantum computer has access to fundamentally different computational resources. One such resource is *entanglement*, and in this exercise, we will see how entanglement can be used to do fun things like *teleportation*.

Suppose that Alice holds a qubit, which we label M, in some unknown, but pure state $|\psi\rangle_M$. In other words,

$$|\psi\rangle_M = \alpha|0\rangle_M + \beta|1\rangle_M, \tag{1}$$

but α and β are unknown to Alice. Her goal is to transmit this state to Bob without ever sending a qubit to Bob and without disturbing the unknown state.

a) Suppose that Alice and Bob each hold half of a maximally entangled pair of qubits in the state

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right). \tag{2}$$

In other words, Alice holds the qubit A, Bob holds the qubit B, and together, they are in the (entangled) state above. Write the total state of the MAB system in the computational basis (i.e. the basis $|0\rangle_M|0\rangle_A|0\rangle_B$, $|0\rangle_M|0\rangle_A|1\rangle_B$, ...).

b) Alice now measures her pair of qubits (MA) in the Bell basis

$$\begin{aligned}|\Phi^\pm\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle \pm |1\rangle|1\rangle\right)\\ |\Psi^\pm\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle \pm |1\rangle|0\rangle\right).\end{aligned} \tag{3}$$

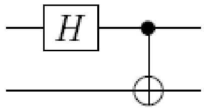
For each of her 4 possible outcomes, what is the state of Bob's qubit?

c) After measuring, Alice is now allowed to send a *classical* message to Bob. What should she tell Bob to do so that he obtains the original state on his qubit, $|\psi\rangle_B$?

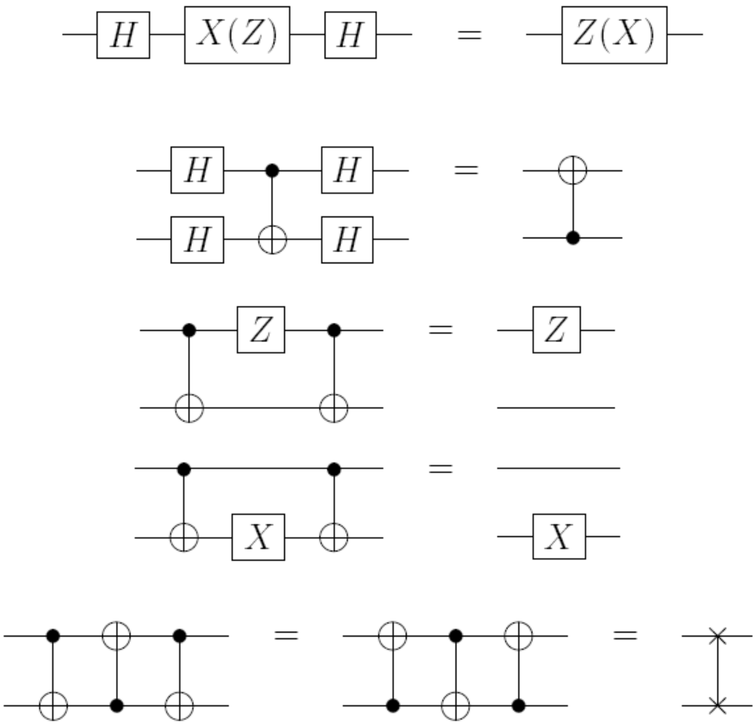
Note – Notice that entanglement let us do something special. Alice never physically sent her quantum message to Bob. Instead, by consuming a shared entangled pair, and by sending a classical messaging consisting of a couple of bits, Alice succeeded in *nonlocally* transmitting her quantum message to Bob. Note that we never violated causality. Can you figure out why?

Problem 2: Quantum gate warm-up

a) Show that the following circuit maps the computational basis to the Bell basis:



b) Demonstrate the following equalities:



(The very last gate denotes the SWAP gate, i.e. $\mathrm{SWAP}|x\rangle|y\rangle = |y\rangle|x\rangle$.)

Problem 3: Distance measures

In many cases, we would like to be able to meaningfully quantify how "close" two quantum states are to each other. For example, a universal gate set has to be able to produce a state that is arbitrarily "close" to any desired output. In this problem, we will see why the 1-norm is a good measure of closeness. For fun, let's do the analysis using density matrices, since we won't talk too much about them in these two lectures and some practice will be useful.

Consider two quantum states described by density operators ρ and $\tilde{\rho}$ in an N-dimensional Hilbert space, and consider the complete orthogonal measurement $\{E_a, a = 1, 2, 3, \ldots, N\}$, where the E_a's are one-dimensional projectors satisfying

$$\sum_{a=1}^{N} E_a = I \tag{4}$$

When the measurement is performed, outcome a occurs with probability $p_a = \mathrm{Tr}\,\rho E_a$ if the state is ρ and with probability $\tilde{p}_a = \mathrm{Tr}\,\tilde{\rho} E_a$ if the state is $\tilde{\rho}$.

The L^1 distance between the two probability distributions is defined as

$$d(p,\tilde{p}) \equiv \|p-\tilde{p}\|_1 \equiv \frac{1}{2}\sum_{a=1}^{N} |p_a - \tilde{p}_a|. \tag{5}$$

This distance is zero if the two distributions are identical, and attains its maximum value of one if the two distributions have support on disjoint sets.

a) Show that

$$d(p,\tilde{p}) \le \frac{1}{2}\sum_{i=1}^{N} |\lambda_i|, \tag{6}$$

where the λ_i's are the eigenvalues of the Hermitian operator $\rho - \tilde{\rho}$. *Hint:* Working in the basis in which $\rho - \tilde{\rho}$ is diagonal, find an expression for $|p_a - \tilde{p}_a|$, and then find an upper bound on $|p_a - \tilde{p}_a|$. Finally, use the completeness property Eq. (4) to bound $d(p,\tilde{p})$.

b) Find a choice for the orthogonal projector $\{E_a\}$ that saturates the upper bound Eq. (6).

Define a distance $d(\rho,\tilde{\rho})$ between density operators as the maximal L^1 distance between the corresponding probability distributions that can be achieved by any orthogonal measurement. From the results of (a) and (b), we have found that

$$d(\rho,\tilde{\rho}) = \frac{1}{2}\sum_{i=1}^{N} |\lambda_i|. \tag{7}$$

c) The L^1 norm $\|A\|_1$ of an operator A is defined as

$$\|A\|_1 \equiv \mathrm{Tr}\left[(AA^\dagger)^{1/2}\right]. \tag{8}$$

How can the distance $d(\rho,\tilde{\rho})$ be expressed as the L^1 norm of an operator?

Now suppose that the states ρ and $\tilde{\rho}$ are pure states $\rho = |\psi\rangle\langle\psi|$ and $\tilde{\rho} = |\tilde{\psi}\rangle\langle\tilde{\psi}|$. If we adopt a suitable basis in the space spanned by the two vectors, and appropriate phase conventions, then these vectors can be expressed as

$$|\psi\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} \qquad |\tilde{\psi}\rangle = \begin{pmatrix} \sin\theta/2 \\ \cos\theta/2 \end{pmatrix}. \tag{9}$$

d) Express the distance $d(\rho,\tilde{\rho})$ in terms of the angle θ.

e) Express $\||\psi\rangle - |\tilde{\psi}\rangle\|^2$ (where $\|\cdot\|$ denotes the Hilbert space norm, i.e., the 2-norm) in terms of θ, and by comparing with the results of (d), derive the bound

$$d(|\psi\rangle\langle\psi|, |\tilde{\psi}\rangle\langle\tilde{\psi}|) \le \||\psi\rangle - |\tilde{\psi}\rangle\|. \tag{10}$$

f) Bob thinks that the norm $\||\psi\rangle - |\tilde{\psi}\rangle\|$ should be a good measure of the distinguishability of the pure quantum states ρ and $\tilde{\rho}$. Explain why Bob is wrong. *Hint:* Remember that quantum states are *rays*.

Problem 4: A universal gate set

If you still have time and are itching for more...

Try out problems 5.3 and 5.4 in this set of notes: http://www.theory.caltech.edu/~preskill/ph219/chap5_15.pdf. These two exercises will step you through a derivation that the Hadamard gate, H, together with a controlled phase gate, $\Lambda(P) \equiv \mathrm{diag}(1, 1, 1, i)$, is a universal gate set.