Assignment 2: Entanglement and ER=EPR

Ph 50 - Fall 2016

Due: 7 November 2016

Please submit your solutions in class or by e-mail to achatwin@caltech.edu

Instructions: Please pick one problem and submit a solution for it, although feel free to take a stab at all three. If you're comfortable with quantum mechanics, I suggest trying Problem 2 or 3.

Problem 1: Two Ways to Black Hole Entropy, Act I

Recall that a black hole is characterized entirely by its mass[1], M. In other words, two black holes of the same mass are physically indistinguishable.

Let's say that you throw some matter into a black hole, which causes the black hole to grow and the matter and its associated entropy to disappear. In order to not violate the second law of thermodynamics, the black hole should itself have some entropy which increases by at least as much as the entropy of the matter you threw in. However, since a black hole is characterized *only* by its mass, the change in entropy can depend only on the mass of what you threw in, and on no other details. In other words, a black hole of mass M must have a higher entropy than any other system of mass M. So, a good way to estimate the entropy of a black hole of mass M is to calculate the highest entropy configuration of ordinary matter of mass M. One particularly high entropy form of "matter" is light.

a) What is the energy (not entropy) of N photons at wavelength λ? Find the equivalent mass M using $E = Mc^2$.

b) For a wide variety of systems, the entropy is roughly proportional to the number of particles. So, the most entropic system with a given mass M would consist of particles of the smallest possible energy (including rest mass). The best candidate would be photons of the greatest possible wavelength (and so lowest energy). Assuming the wavelength λ can be no greater than the radius R of the black hole, use the approximation $S \approx k_B N$ to find the greatest possible entropy of a system of photons of equivalent mass M.

c) Using the relationship $R = 2GM/c^2$, show that the entropy in part b) is proportional to the surface area of the black hole.

Note that this is a pretty good estimate—the actual entropy of a black hole is

$$S_{BH} = \frac{k_B c^3 A}{4G\hbar}\,. \tag{1}$$

Problem 2: Two Ways to Black Hole Entropy, Act II

Alternatively, recall the model that we suggested in class for a two-sided black hole: $2N$ pairs of entangled spins, where N spins are associated to each event horizon. The reduced state of the N spins associated to a single event horizon is therefore a mixed state with some von Neumann entropy.

[1] A black hole can also have charge and angular momentum, but let's just consider an uncharged, non-rotating black hole.

a) First, consider a pair of spins in the state

$$|\psi^-\rangle_{12} = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2 \right) . \tag{2}$$

Correspondingly, the density matrix is $\rho = |\psi^-\rangle_{12}\langle\psi^-|_{12}$.

i) What is the reduced density matrix of spin 1 that is obtained by tracing out spin 2, i.e.,

$$\rho_1 \equiv \mathrm{tr}_2\rho = \langle\uparrow|_2\ \rho\ |\uparrow\rangle_2 + \langle\downarrow|_2\ \rho\ |\downarrow\rangle_2\,? \tag{3}$$

ii) What is the von Neumann entropy of spin 1, $S = -\mathrm{tr}\ \rho_1 \log \rho_1$?

b) Now consider a collection of $2N$ spins in the state

$$|\Psi\rangle = |\psi^-\rangle_{11'} \otimes |\psi^-\rangle_{22'} \otimes \cdots \otimes |\psi^-\rangle_{NN'}\,, \tag{4}$$

i.e., the spins are all paired up into Bell pairs.

i) What is the reduced density matrix that is obtained by tracing out the second spin in each pair (spins $1'$, $2'$, ..., N'), i.e.,

$$\tilde{\rho} \equiv \mathrm{tr}_{1'\cdots N'}|\Psi\rangle\langle\Psi|\,? \tag{5}$$

ii) What is the von Neumann entropy of $\tilde{\rho}$, $S = -\mathrm{tr}\ \tilde{\rho}\log\tilde{\rho}$?

iii) If you equate this von Neumann entropy with the Beckenstein-Hawking entropy given in Eq. (1) above, what is N as a function of the black hole mass, M? How does this compare to the thermodynamic estimate in Problem 1?

Problem 3: How Robust is Purity?

ER=EPR is the bold claim that entanglement and wormholes are intimately related. In this problem, we will continue to use our $2N$ spins model for a two-sided black hole with a wormhole, and we will investigate the importance of entanglement alongside other properties of the state of the spins.

Following the thesis that entanglement and wormholes are related, we indeed considered an entangled state of $2N$ spins in Problem 2. However, this state is very special in that it factorizes into Bell pairs. As such, we shouldn't be convinced that entanglement is really the "special" thing here; maybe the fine-tuning of the state is what's important, and entanglement just comes along for the ride.

It turns out, however, that *random* states are actually highly entangled in general when you partition them into two sets of equal size. As such, let's instead consider a very general state of $n \equiv 2N$ spins:

$$|\Psi\rangle = \sum_{x_1\cdots x_n} c_{x_1\cdots x_n}|x_1 \ldots x_n\rangle \tag{6}$$

In the above, $x_1 \cdots x_n$ denotes a n-bit string of zeros and ones, and the coefficients $c_{x_1\cdots x_n}$ are a random amplitudes that square-sum to 1, i.e.,

$$\langle\,|c_{x_1\cdots x_n}|\,\rangle = \frac{1}{\sqrt{2^n}} \qquad \text{and} \qquad \frac{c_{x_1\cdots x_n}}{|c_{x_1\cdots x_n}|} \sim e^{i\phi(x_1\cdots x_n)}, \tag{7}$$

where each $\phi(x_1 \cdots x_n)$ is a random phase drawn from the uniform distribution on $(0, 2\pi)$.

Nevertheless, how can we convince ourselves that entanglement really is the relevant generic characteristic of this analogue wormhole state? For instance, could it be that purity of the state $|\Psi\rangle$ is important?

a) If n is very large, we shouldn't expect much to change about the wormhole if we remove a single spin, e.g. by Hawking evaporation. As such, lets see what happens when we trace out a single spin. What is the reduced density matrix

$$\tilde{\rho} = \mathrm{tr}_1 |\Psi\rangle\langle\Psi| \tag{8}$$

of the $n-1$ spins when you trace out the first spin? *Hint*: I recommend using the notation

$$\langle 0_1|\Psi\rangle = \sum_{x_2\cdots x_n} c_{0x_2\cdots x_n}|x_2\cdots x_n\rangle \equiv \sum_x c_{0x}|x\rangle$$
$$\langle 1_1|\Psi\rangle = \sum_{x_2\cdots x_n} c_{1x_2\cdots x_n}|x_2\cdots x_n\rangle \equiv \sum_x c_{1x}|x\rangle$$

for clarity and compactness (i.e. abbreviate a $(n-1)$-bit string by x).

b) The *purity* of the state $\tilde{\rho}$ is defined as $\mathrm{tr}\,\tilde{\rho}^2$, and it is equal to 1 if the state is pure. What is the purity of the state $\tilde{\rho}$ that you found in part a) in terms of the coefficients c_{0x} and c_{1x}?

c) For large n and random coefficients as described above, estimate the purity. *Hint*: you may find the following result useful:

Lemma 0.1 *Let $\phi_1, \ldots, \phi_m$ be uniform random variables on the interval $(0, 2\pi)$. Then, it follows that*

$$\left\langle \left| \sum_{a=1}^{m} e^{i\phi_a} \right|^2 \right\rangle = m$$

What does this say about purity? Is the purity of $|\Psi\rangle$ likely important for ER=EPR?