Problem 1: Annealing Ice Cream

No, it's not a good idea to anneal ice cream. The smoothness and creaminess of ice cream comes from the fact that, at the molecular level, the water-ice crystals that are suspended in the ice cream are on average very small and disordered. If you melt ice cream and then let it cool down slowly, large ice crystals are able to form, which produce hard and chunky ice cream. This is also why the use of liquid nitrogen is often touted as a great way to make ice cream: it causes the cream to cool down so quickly that large ice crystals do not have time to form.

Problem 2: Annealing a Single Spin

(a) From Eq. (*5*), we have that

$$\beta(t) = g(t)^{-1}(-i\hbar\dot{\alpha}(t) - J\alpha(t)) . \tag{1}$$

Next, taking a derivative with respect to t of both sides of Eq. (*5*), we obtain

$$\begin{aligned} i\hbar\ddot{\alpha}(t) &= -J\dot{\alpha}(t) - \dot{g}(t)\beta(t) - g(t)\dot{\beta}(t) && (2)\\ &= -J\dot{\alpha}(t) - \frac{\dot{g}(t)}{g(t)}(-i\hbar\dot{\alpha}(t) - J\alpha(t)) - \frac{g(t)}{i\hbar}\left(-g(t)\alpha(t) + \frac{J}{g(t)}(-i\hbar\dot{\alpha}(t) - J\alpha(t))\right) . && (3) \end{aligned}$$

In the last line we used Eq. (*6*) to express $\dot{\beta}(t)$ in terms of α and $\dot{\alpha}$. After some rearrangement, one ultimately obtains

$$\ddot{\alpha}(t) - \frac{\dot{g}(t)}{g(t)}\dot{\alpha}(t) + \frac{1}{\hbar^2}\left(i\hbar J\frac{\dot{g}(t)}{g(t)} + g(t)^2 + J^2\right)\alpha(t) = 0 . \tag{4}$$

According to Eq. (*5*), given an initial condition $\{\alpha(0), \beta(0)\}$, the initial condition for $\dot{\alpha}(t)$ is

$$\dot{\alpha}(0) = -\frac{1}{i\hbar}(J\alpha(0) + g(0)\beta(0)) . \tag{5}$$

(b) For $g(t) = g_0 e^{-Jt/\hbar}$, it follows that $\dot{g}/g = -J/\hbar$, so

$$\ddot{\alpha}(t) + \frac{J}{\hbar}\dot{\alpha}(t) + \frac{J^2}{\hbar^2}\left(1 - i + \frac{g_0^2}{J^2}e^{-2Jt/\hbar}\right)\alpha(t) = 0 . \tag{6}$$

(c) Let $x = (g_0/J)e^{-Jt/\hbar}$, so $dx/dt = -(J/\hbar)x$. Note that

$$\begin{aligned} \frac{d\alpha}{dt} &= \frac{d\alpha}{dx}\frac{dx}{dt} = -\frac{J}{\hbar}x\frac{d\alpha}{dx} && (7)\\ \frac{d^2\alpha}{dt^2} &= -\frac{J}{\hbar}\frac{d}{dx}\left(x\frac{d\alpha}{dx}\right)\frac{dx}{dt} = \frac{J^2}{\hbar^2}x\left(\frac{d\alpha}{dx} + x\frac{d^2\alpha}{dx^2}\right) . && (8) \end{aligned}$$

Therefore, plugging Eqs. (7) and (8) into (6), one finds that

$$x^2\alpha''(x) + (1 - i + x^2)\alpha(x) = 0 , \tag{9}$$

where a prime denotes differentiation with respect to x.

(d) Eq. (9) is the Liouville form of Bessel's differential equation; the general solution is

$$\alpha(x) = C_1\sqrt{x}J_{\frac{1}{2}+i}(x) + C_2\sqrt{x}Y_{\frac{1}{2}+i}(x) . \tag{10}$$

(e) The probability $|\alpha(x)|^2$ is given by

$$|\alpha(x)|^2 = |C_1|^2\, x\, J_{\frac{1}{2}+i}(x) J_{\frac{1}{2}-i}(x) + |C_2|^2\, x\, Y_{\frac{1}{2}+i}(x) Y_{\frac{1}{2}-i}(x) + C_1^* C_2\, x\, J_{\frac{1}{2}-i}(x) Y_{\frac{1}{2}+i}(x) + C_1 C_2^*\, x\, J_{\frac{1}{2}+i}(x) Y_{\frac{1}{2}-i}(x)\,. \tag{11}$$

Using the asymptotic forms of the Bessel J and Y functions given in Eq. (*8*) and taking a limit as $x \to 0^+$, one finds that

$$\lim_{x\to 0^+} |\alpha(x)|^2 = \frac{2|C_2|^2}{\pi \cosh \pi} \tag{12}$$

(f) From the above, we see that the probability to wind up in the desired ground state—which is given by $|\alpha|^2$—depends on the magnitude of C_2. As such, we should work out C_2 for the initial conditions $\alpha(t=0) = \beta(t=0) = 1/\sqrt{2}$. From the definition of x, it follows that $x(t=0) = g_0/J$. Next, since $d\alpha/dt = -(J/\hbar) x d\alpha/dx$, it follows that

$$\left.\frac{d\alpha}{dx}\right|_{x=g_0/J} = -\frac{\hbar}{J}\,\frac{1}{x(t)}\,\left.\frac{d\alpha}{dt}\right|_{t=0} = \frac{1+(J/g_0)}{i\sqrt{2}}, \tag{13}$$

where we used the result of Eq. (5) and the definition of g to work out the initial condition $\dot{\alpha}(0)$. Therefore, letting $\epsilon = J/g_0$, the initial condition for $\alpha(x)$ is

$$\alpha(1/\epsilon) = \frac{1}{\sqrt{2}} \qquad \alpha'(1/\epsilon) = \frac{1+(J/g_0)}{i\sqrt{2}}\,. \tag{14}$$

For the general solution (10), in particular this gives that

$$C_2 = \sqrt{\frac{\epsilon}{2}}\,\frac{iJ_{\frac{1}{2}+i}(1/\epsilon) + J_{-\frac{1}{2}+i}(1/\epsilon)}{J_{-\frac{1}{2}+i}(1/\epsilon) Y_{\frac{1}{2}+i}(1/\epsilon) - Y_{-\frac{1}{2}+i}(1/\epsilon) J_{\frac{1}{2}+i}(1/\epsilon)}\,. \tag{15}$$

So, what does this mean for the probability $|\alpha|^2$? Lets first consider the regime where $g_0 \gg J$, *i.e.*, the applied field is much larger than the constant weak field and it decays slowly compared to the Larmor frequency of the weak field. In this regime $\epsilon \ll 1$, so $1/\epsilon \gg 1$ and we should use the large-argument pseudo-asymptotic expansions of the Bessel J and Y functions to work out C_2. Using Eq. (*9*), one finds that

$$\lim_{\epsilon\to 0} |C_2|^2 = \frac{\pi}{4(\cosh(\pi/2) - \sinh(\pi/2))^2}\,, \tag{16}$$

for which

$$\lim_{x\to 0^+} |\alpha(x)|^2 \xrightarrow{\epsilon\to 0} \frac{1}{2\cosh(\pi)(\cosh(\pi/2) - \sinh(\pi/2))^2} \approx 0.998 \tag{17}$$

So, in this regime we obtain the correct ground state with very high probability.

We could also consider the regime in which $g_0 \ll J$, *i.e.*, we have a small applied field which decays rapidly. Expanding the Bessel functions for small $1/\epsilon$, one finds that

$$|C_2|^2 = \frac{\pi \cosh \pi}{4} + \frac{\pi \cosh \pi}{5}\epsilon^{-1} + O(\epsilon^{-2}), \tag{18}$$

and so

$$\lim_{x\to 0^+} |\alpha(x)|^2 = \frac{1}{2} + \frac{2}{5\epsilon} + O(\epsilon^{-2})\,. \tag{19}$$

As one would expect, annealing performs quite poorly in this case!

As an illustration of the preceding discussion, a plot of $|\alpha(x\to 0^+)|^2$ is shown in Fig. 1 below.

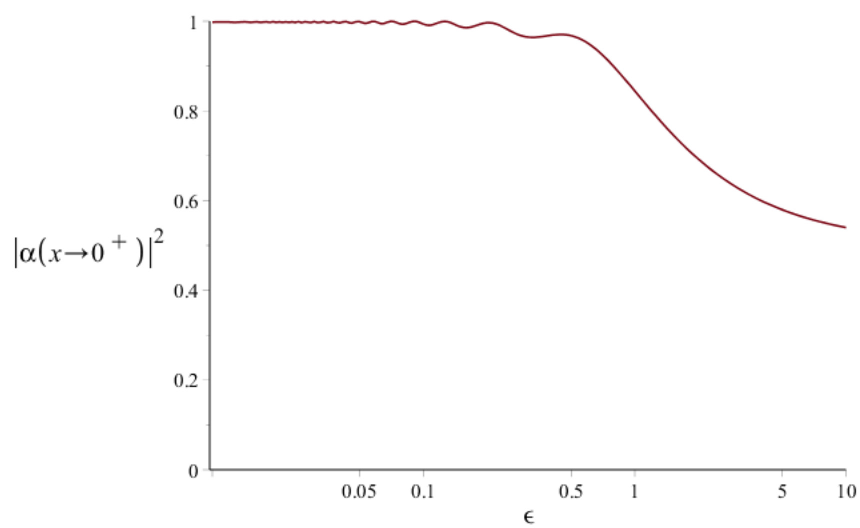

Figure 1: Plot of $|\alpha(x \to 0^+)|^2$ as a function of $\epsilon = J/g_0$.