GAPS IN THE RINGS OF SATURN

Did you know that Saturn is the most quantum planet? The symbol for Saturn is just a stylized version of the reduced Planck constant!

$$\hbar \tag{1}$$

Typography aside, Saturn is perhaps most notable for its magnificent rings, which (visibly) extend out to about 140,000 km from the centre of the planet. Yet, despite their considerable extent, the rings are only about 10-100 km thick.

The composition of the rings, their radial extent, and how they got there in the first place are all interesting questions, but here, let's investigate the band structure of the rings. Upon closer examination (with, e.g., the Cassini probe), one quickly realizes that the rings are not one homogeneous disc, but rather are highly striated and filled with gaps. An artist's rendition of the gap structure is shown below in Fig. 1, indicating some of the most prominent gaps in the rings.

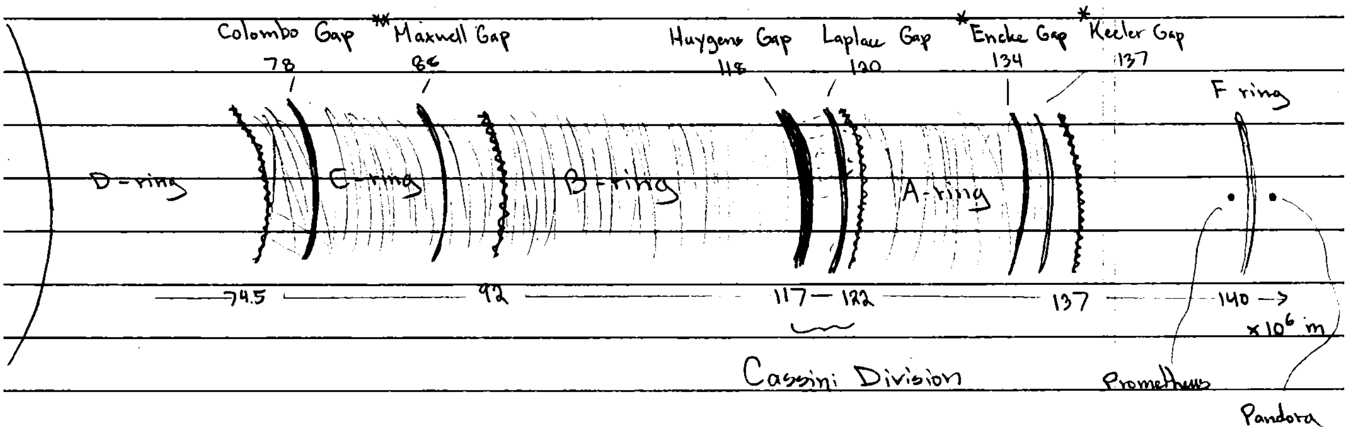

FIG. 1. Sketch of Saturn's D, C, B, A, and F rings. Boundaries between the rings are indicated by rough jagged lines. Some of the most prominent gaps in the rings, as well as their radial locations, are shown.

What causes these gaps? Some gaps, such as the Encke Gap and Keeler Gap, harbour "trash collector" moonlets that clear out their orbit of debris. The F-ring is held together by two "shepherd moons," Prometheus and Pandora, that confine the ring to a space in between their orbits. Then, there are gaps that are caused by orbital resonances with Saturn's moons.

An orbital resonance occurs between two circular orbits when the period of one orbit is a rational multiple of the other. When this happens, bodies in the two orbits will keep meeting up at the same angular position over and over again. Now suppose that the outer orbit is occupied by some large body, like a moon. Whenever a small body in the inner resonant orbit meets up with the moon, it will feel a minuscule tug from the moon, but this effect coherently builds up over time and eventually destabilizes the inner body's orbit—it becomes elliptical (Fig 2). If you're in an elliptical orbit, however, at the apogee you will be moving slower than the other bodies around you that are in a circular orbit. Therefore, collisions with these bodies drag you forward and eventually kick you into a new higher circular orbit. In this way, moons can clear out gaps in the rings via orbital resonances.

So, is this a reasonable explanation? Let's look at some data. The table below lists some properties of Saturn's seven largest moons (which together make up 99% of the mass orbiting Saturn), including their orbital period and some resonant orbits.

Moon	mass ($\times 10^{18}$ kg)	orbit ($\times 10^6$ m)	T (Earth days)	$R_{2:1}$ ($\times 10^6$ m)	$R_{3:1}$ ($\times 10^6$ m)	$R_{3:2}$ ($\times 10^6$ m)
Titan	134000	1222	16	770	587	993
Rhea	2310	527	4.5	332	253	402
Iapetus	1810	3561	79	2243	1712	2718
Dione	1100	377	2.7	237	181	288
Tethys	617	295	1.9	186	142	225
Enceladus	108	238	1.4	150	114	182
Mimas	38	185	0.9	117	89	141

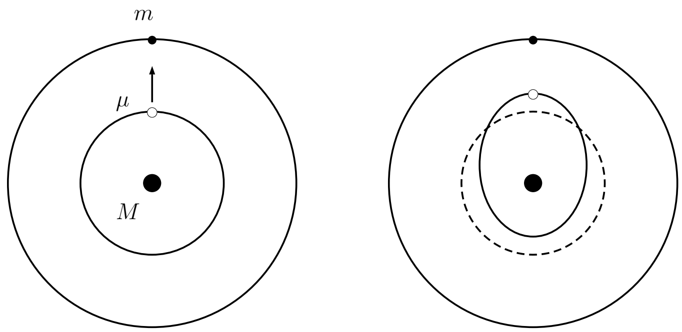

FIG. 2. A small body of mass μ gets perturbed by the moon of mass m.

The largest moon for which an orbit with half its period lies inside the ring is Mimas, with $R_{2:1} = 117 \times 10^6$ m. This is pretty much the location of the Huygens gap, and indeed, it is claimed that the gap is due to a resonance with Mimas!

What about higher resonances, for example a 3:1 period ratio or a 4:1 ratio, where the resonant orbit is closer to Saturn? Referring to the 3:1 column in the table, there don't appear to be any evident structures at $R_{3:1}$ for Mimas or Enceladus. As you move closer to Saturn, the tug from the moons gets weaker, but since you're orbiting faster, you have resonant "encounters" with the moons more often. Judging from the data, it seems that the strength of the encounter is more important than the frequency—let's try to analyze this.

A. The effect of a single encounter

Consider the following simple setup. At the centre of our reference frame we have Saturn, with mass M, and at some radius R_{moon}, one of its moons orbits in a circular orbit. Consider now a test mass μ which orbits at $R < R_{\text{moon}}$, and let's suppose that its orbital period is a rational multiple of T_{moon}, i.e.,

$$\frac{T}{T_{\text{moon}}} = \frac{k}{n} \qquad k, n \in \mathbb{N},\ k < n. \tag{2}$$

Let's work in a reference frame that rotates with the moon so that the moon appears stationary at $\theta = 0$ and the test mass rotates with an angular speed equal to

$$\omega_{\text{eff}} = \omega - \omega_{\text{moon}} = 2\pi \left(\frac{1}{T} - \frac{1}{T_{\text{moon}}} \right) > 0\,. \tag{3}$$

To model the effect that the moon has on the test mass, let's suppose that the interactions are dominated by the "encounters" that happen whenever the test particle is at $\theta = 0$. In particular, what I'm proposing is that we quantify the effect by the following impulse:

$$\mathbf{I} = \int_{\Delta\theta} \mathbf{F}_{\text{moon}}\, dt \approx F_{\text{moon}}(\theta = 0) \frac{\Delta\theta}{\omega_{\text{eff}}}\, \hat{\mathbf{x}} \tag{4}$$

Here, $\mathbf{F}_{\text{moon}}$ is the force that the moon exerts on the test mass, and $\Delta\theta$ is some small angular wedge (Fig. 3). This should be a reasonable model provided that the test mass is far enough away from the moon, or in other words, that encounters aren't prolonged events. Moreover, it's reasonable that there should be some minimum distance that must separate the moon from the test mass, since at some point the moon acts like a trash collector.

F_{moon} is just given by Newton's law,

$$F_{\text{moon}} = \frac{Gm\mu}{(R_{\text{moon}} - R)^2}, \tag{5}$$

so plugging this into $\mathbf{I}$ and using the expression (3) for ω_{eff}, we find that

$$|\mathbf{I}| = \frac{\Delta\theta}{2\pi} \frac{Gm\mu}{R_{\text{moon}}^2 / T_{\text{moon}}} \frac{1}{(T_{\text{moon}}/T - 1)\,(1 - R/R_{\text{moon}})^2}\,. \tag{6}$$

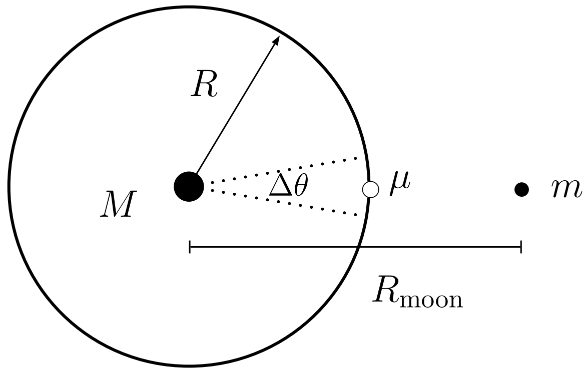

FIG. 3. Model the effect of the moon as discrete impulses when the test mass is nearby.

Noticed that I pulled out a factor of $R_{\rm moon}^2/T_{\rm moon}$ in the denominator. Next let's use the relation $T/T_{\rm moon} = k/n$, and we also know that

$$\frac{R}{R_{\rm moon}} = \left(\frac{T}{T_{\rm moon}}\right)^{2/3} = \left(\frac{k}{n}\right)^{2/3}, \tag{7}$$

so

$$|\mathbf{I}| = \frac{\Delta\theta}{2\pi} \frac{Gm\mu}{R_{\rm moon}^2/T_{\rm moon}} \frac{1}{(n/k-1)\left(1-(k/n)^{2/3}\right)^2}. \tag{8}$$

The prefactors in Eq. (6) are just a constant impulse, so the most important part is the numerical factor

$$\tilde{w}(k/n) \equiv \frac{1}{(n/k-1)\left(1-(k/n)^{2/3}\right)^2}. \tag{9}$$

This tells us how the strength of a single encounter depends on the orbital ratio k/n.

B. The effect of multiple encounters over time

Next, let's account for the fact that different orbital ratios have different encounter rates. The first question that we want to answer is: Over k periods of the moon, how many encounters occur?

Suppose the test particle begins at $\theta = 0$ at $t = 0$. Then its angular position over time will be

$$\theta(t) = \omega_{\rm eff} t. \tag{10}$$

Encounters happen when $\theta(t) = 2\pi j$ for $j \in \mathbb{N}$. Solving for t, we find that

$$t_j = \frac{j}{n/k-1} T_{\rm moon}. \tag{11}$$

If we count the encounter at $t = 0$ as the first encounter, then $t = kT_{\rm moon}$ will be the start of a new cycle, which happens when

$$k = \frac{j}{n/k-1} \quad \Rightarrow \quad j = n-k. \tag{12}$$

Therefore, during k periods of the moon, $n-k$ encounters occur.

Consequently, let's define the normalized numerical factor

$$w(k/n) = \frac{n-k}{k}\tilde{w}(k/n) = \frac{1}{\left(1-(k/n)^{2/3}\right)^2}. \tag{13}$$

This tells us how the moon affects a test mass with orbital ratio k/n, normalized per unit time. The factor $w(k/n)$ blows up as $k/n \to 1$, and so it seems like being closer to the moon is favoured.

C. Discussion

The result above seems to explain why we see 2:1 resonances, but not higher simple resonances. But, what about other resonances like 17:11? The rationals are dense in the real numbers after all, so why are there rings at all?

I think that the answer is that it's also important *where* the encounters occur in the *nonrotating* frame. For example, if the moon and a test mass with a 2:1 resonance have an encounter at $\theta = 0$, then the encounter keeps happening at $\theta = 2\pi j = 0 \pmod{2\pi}$. Since the rationale is that the moon disturbs the test mass' circular orbit and pushes it into an elliptical orbit, it's important that the perturbation keeps happening at the same location so that the test mass' orbit gets more and more deformed. On the other hand, for general n/k, encounters happen at

$$\theta_j = 2\pi \frac{j}{n/k - 1}, \tag{14}$$

so while encounters at a given angle will eventually start repeating, the test mass receives a lot of kicks at different angular positions before this happens.

One possible issue with this explanation is that it means that a 3:2 resonance should also have a big effect; the encounter always happens at $\theta = 0$. However, looking back at the table, $R_{3:2}$ for Mimas happens... precisely where the F-ring is located! It could be that the effect of the shepherd moons is enough to dominate over the orbital resonance effects, but either way, it seems like orbital resonances alone are not the end of the story.

D. Exercise

In the analysis above, we tacitly assumed that the test mass orbits in the same direction as the moon. What if the test mass orbits in the opposite direction, i.e., in a *retrograde* orbit? What is $w(k/n)$ in this case?

E. A bonus challenge

We also simply assumed that orbital resonances were indeed a destabilizing mechanism. It should be possible to derive this. Feel free to attack this challenge however you see fit; however, here is what I think is a promising line of attack in case you want some inspiration.

Assume that the test mass μ is so small that it doesn't affect the orbit of the moon, which moves in a perfect circular orbit around Saturn. In the co-rotating frame, the forces acting on the test mass are

$$\mathbf{F} = \mathbf{F}_{♄} + \mathbf{F}_{\text{moon}} + \mathbf{F}_{\text{centrifugal}} + \mathbf{F}_{\text{Coriolis}} \tag{15}$$

$$= -GM\mu \frac{\mathbf{r}}{|r|^3} - Gm\mu \frac{(\mathbf{r} - \mathbf{R}_{\text{moon}})}{|\mathbf{r} - \mathbf{R}_{\text{moon}}|^3} - \mu \boldsymbol{\omega}_{\text{moon}} \times (\boldsymbol{\omega}_{\text{moon}} \times \mathbf{r}) - 2\mu \boldsymbol{\omega}_{\text{moon}} \times \dot{\mathbf{r}} \tag{16}$$

$\mathbf{r}$ is the position of the test mass, and in the diagram below, I've chosen $\mathbf{R}_{\text{moon}} = R_{\text{moon}}\hat{\mathbf{x}}$ and $\boldsymbol{\omega}_{\text{moon}} = \omega_{\text{moon}}\hat{\mathbf{z}}$. Were it not for $\mathbf{F}_{\text{moon}}$, the test mass would move on the circular orbit

$$\mathbf{r}_0(t) = (x_0(t), y_0(t)) = (R\cos\omega_{\text{eff}}t, R\sin\omega_{\text{eff}}t)\,. \tag{17}$$

My suggestion is as follows: Write down Newton's equation for the test mass $\mathbf{F} = \mu\ddot{\mathbf{r}}$, expand $\mathbf{r}(t)$ in a perturbation series,

$$\mathbf{r}(t) = \mathbf{r}_0(t) + \mathbf{r}_1(t) + \dots \tag{18}$$

and find the equation of motion for $\mathbf{r}_1(t)$. I suspect that it will become perturbatively unstable when there is an orbital resonance.

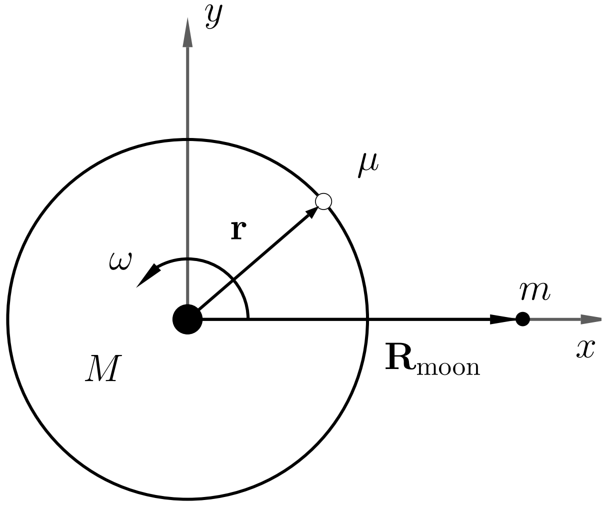

FIG. 4. Set-up for the bonus challenge.