Lecture 2 - Quantum Channels

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Def^n A quantum channel is a map that sends states to states.

Why channels? - dynamics is interesting!
- beyond unitary evolution
- universal — every quantum process is a channel

Ex Unitary evolution: Let $\rho \in S(\mathcal{H})$, $U \in \mathcal{U}(\mathcal{H})$.

$$\mathcal{N} : S(\mathcal{H}) \to S(\mathcal{H}) \qquad \text{is a channel}$$
$$\rho \mapsto U \rho U^\dagger$$

Ex Nonunitary evolution: Let $\rho_A \in S(\mathcal{H}_A)$, $|0\rangle_B \in \mathcal{H}_B$ some fixed state

$U_{AB} \in \mathcal{U}(\mathcal{H}_{AB})$. $\quad \mathcal{N} : S(\mathcal{H}_A) \to S(\mathcal{H}_A)$

$$\rho_A \mapsto \mathrm{Tr}_B(U_{AB}\, \rho_A \otimes |0\rangle\langle 0|_B\, U_{B}^\dagger)$$

Also a channel!

Q: what properties should channels have?

Let $\mathcal{N} : S(\mathcal{H}_A) \to S(\mathcal{H}_B)$...

1. Trace-preserving : $\mathrm{Tr}_B[\mathcal{N}(\sigma)] = \mathrm{Tr}_A[\sigma]$

2. Linear $\mathcal{N}(\lambda_1 \rho + \lambda_2 \sigma) = \lambda_1 \mathcal{N}(\rho) + \lambda_2 \mathcal{N}(\sigma)$

Why? $\rightsquigarrow$ Consistency w/ ensemble interp. of density mat.

$\rho = \sum p_i \rho_i$ if prepare ρ_i w/ prob. p_i, then after acting w/ $\mathcal{N}$, should get $\mathcal{N}(\rho_i)$ w/ prob. p_i

Ex $\mathcal{E}(\rho) = e^{i\pi\sigma_x \mathrm{Tr}[\sigma_x \rho]}\, \rho\, e^{-i\pi\sigma_x \mathrm{Tr}[\sigma_x \rho]}$

Scenario 1: prepare $\rho_1 = \frac{1}{2}|\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2}|\downarrow_z\rangle\langle\downarrow_z|$

$\mathrm{Tr}[\sigma_x \rho_1] = 0 \Rightarrow \mathcal{E}(\rho) = 0$

Scenario 2: Prepare ρ_1, then put through apparatus: if $|\downarrow_z\rangle$, flip to $|\uparrow_x\rangle$ $\longrightarrow$ $\rho_2 = \frac{1}{2}|\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2}|\uparrow_x\rangle\langle\uparrow_x|$

$\mathrm{Tr}[\sigma_x \rho_2] = \frac{1}{2} \Rightarrow \mathcal{E}(\rho_2) = \sigma_x \rho_2 \sigma_x = \frac{1}{2}|\downarrow_z\rangle\langle\downarrow_z| + \frac{1}{2}|\uparrow_x\rangle\langle\uparrow_x|$

$\rightarrow$ $|\uparrow_z\rangle\langle\uparrow_z|$ evolves differently depending on how we (would have) prepared in the other case — weird!

(3-ε) Positive: if X_A is pos. semidef, then $N(X_A)$ should also be pos. semidef.

// recall, pos. semidef $\equiv \langle\psi|X|\psi\rangle \geq 0 \ \forall |\psi\rangle \Leftrightarrow \mathrm{spec}\, X \subseteq [0,\infty)$

- actually, require something a bit stronger...

3. Completely Positive: Given any other $\mathcal{H}_R$, require that $\mathrm{id}_R \otimes N$ positive on $\mathcal{L}(\mathcal{H}_{RA})$

- physically reasonable; even if a channel only acts on a part of the universe, it should map a state of the universe to a state of the universe

Ex Transpose is positive... $T: |i\rangle\langle j| \longmapsto |j\rangle\langle i|$

$\langle\psi|\rho^T|\psi\rangle = \sum_{ij} \psi_i^* (\rho^T)_{ij} \psi_j = \sum_{ij} \psi_j \rho_{ji} \psi_i^* = \langle\psi^*|\rho|\psi^*\rangle \geq 0$

... but not CP

- extend to $\mathcal{H}_B$, consider action on $|\tilde{\Phi}\rangle_{AB} = \sum_{i=0}^{d-1} |i\rangle_A |i\rangle_B$

$$(T_A \otimes \mathrm{id}_B)(|\tilde{\Phi}\rangle\langle\tilde{\Phi}|_{AB}) = (T_A \otimes \mathrm{id}_B)\left(\sum_{ij} |i\rangle\langle j|_A \otimes |i\rangle\langle j|_B\right)$$
$$= \sum_{ij} |j\rangle\langle i|_A \otimes |i\rangle\langle j|_B$$
$$= \mathrm{SWAP}_{AB}$$

but $(\mathrm{SWAP})^2 = I \Rightarrow$ eigenvals of SWAP are ± 1 $\rightarrow$ not pos.

Def^n A quantum channel is a map $\mathcal{N} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ that is linear, trace-preserving, and completely positive

$\exists$ unique ext. from $\mathcal{S}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A)$
see, eg. Wilde App. B

Thm (Choi-Kraus) A linear map $\mathcal{N} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ is CPTP iff

$$\mathcal{N}(X_A) = \sum_{l=0}^{d-1} M_l X_A M_l^\dagger \qquad \text{——} (*)$$

$\forall\ X_A \in \mathcal{L}(\mathcal{H}_A)$, where $M_l \in \mathcal{L}(\mathcal{H}_A, \mathcal{H}_B)$: $\sum_{l=0}^{d-1} M_l^\dagger M = I_A$, may be chosen such that $d \leq d_A d_B$

Note: $(*)$ is the operator-sum expansion of $\mathcal{N}$. The ops. V_l are called Kraus operators.

ex
$$\begin{aligned}
\mathcal{N}(\rho_A) &= \mathrm{Tr}_B\left[U_{AB}\, \rho_A \otimes |0\rangle\langle 0|_B\, U_{AB}^\dagger \right] \\
&= \sum_{j=0}^{d_B-1} {}_B\langle j| U_{AB} |0\rangle_B\, \rho_A\, {}_B\langle 0| U_{AB}^\dagger |j\rangle_B \\
&\equiv \sum_j M_j \rho_A M_j^\dagger
\end{aligned}$$

where $M_j \equiv {}_B\langle j| U_{AB} |0\rangle_B$

check:
$$\begin{aligned}
\sum_j M_j^\dagger M_j &= \sum_j {}_B\langle 0| U_{AB}^\dagger |j\rangle\langle j| U_{AB} |0\rangle_B \\
&= {}_B\langle 0| U_{AB}^\dagger \Big(\underbrace{\textstyle\sum_j |j\rangle\langle j|_B}_{I_B} \Big) U_{AB} |0\rangle_B \\
&= {}_B\langle 0| I_{AB} |0\rangle_B \\
&= I_A
\end{aligned}$$

Important consequence: isometric dilation
Given $\mathcal{N}_{A\to B}$, let $\mathcal{H}_E$: $\dim \mathcal{H}_E \geq d$ (as above). Then, $\exists$ a linear isometry $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$

$$\mathrm{Tr}_E\left[V X_A V^\dagger \right] = \mathcal{N}_{A\to B}(X_A), \quad V^\dagger V = I_A, \quad VV^\dagger = \Pi_{BE}$$

Notes:

- isometry ≡ inner prod preserving linear map. i.e. $\langle V\phi | V\psi \rangle = \langle \phi | \psi \rangle$
- easy to extend isometry to a unitary by adding aux. Hilbert spaces

  e.g. $\mathcal{H}_A = \mathrm{span}\{|0\rangle_A\}$ $\quad V: |0\rangle_A \longmapsto |0\rangle_B + |1\rangle_B$ is a isom.

  $\mathcal{H}_B = \mathrm{span}\{|0\rangle_B, |1\rangle_B\}$

  unitary extension: let $\mathcal{H}_{A'} = \{|1\rangle_{A'}\}$, $\quad U: \mathcal{H}_A \oplus \mathcal{H}_{A'} \longrightarrow \mathcal{H}_B$

  - e.g. let $U: |0\rangle_A \longmapsto |0\rangle_B + |1\rangle_B$ $\quad U|_A = V$

    $|1\rangle_{A'} \longmapsto |0\rangle_B - |1\rangle_B$

- Π_{BE} ≡ projector onto $V(\mathcal{L}(\mathcal{H}_A))$
  - must be a projector because $VV^\dagger VV^\dagger = V\underbrace{(V^\dagger V)}_{I_A}V^\dagger = VV^\dagger$

- easy to construct $V = \sum_{j=0}^{d-1} M_j \otimes |j\rangle_E$

check:
$$\begin{aligned} V^\dagger V &= \textstyle\sum_{ij} M_i^\dagger M_j \langle i|j\rangle_E \\ &= \textstyle\sum_i M_i^\dagger M_i \\ &= I_A \quad \checkmark \end{aligned}$$

$$\begin{aligned} \mathrm{Tr}_E[VX_AV^\dagger] &= \mathrm{Tr}_E\left[\textstyle\sum_{ij} M_i X_A M_j^\dagger \otimes |i\rangle\langle j|_E\right] \\ &= \textstyle\sum_{ij} M_i X_A M_j^\dagger \langle j|i\rangle_E \\ &= \textstyle\sum_i M_i X_A M_i^\dagger \\ &= \mathcal{N}(X_A) \end{aligned}$$

- Kraus ops are not unique

  e.g. if $|j\rangle_E = \sum_\mu W_{j\mu} |\mu\rangle_E$ (unitary change of basis),

$$\begin{aligned} V &= \textstyle\sum_{j=0}^{d-1} M_j \otimes (\sum_\mu W_{j\mu} |\mu\rangle_E \\ &= \sum_\mu \left(\sum_j W_{j\mu} M_j\right) \otimes |\mu\rangle_E \equiv \sum_\mu N_\mu \otimes |\mu\rangle_E \end{aligned}$$

(always related by unitary like this if same channel — see, e.g. Wilde)

Recap: - any channel has operator-sum expansion

- can always think of as unitary evo^n in larger $\mathcal{H}$

Proof of Choi-Kraus: ($\Rightarrow$) this is the easy direction.

- suppose action of N given by $(*)$
- clearly linear
- CP? $(id_R \otimes N_{A\to B})(X_{RA}) = \sum_\ell (I_R \otimes M_\ell) X_{RA} (I_R \otimes M_\ell^\dagger)$

$${}_{RB}\langle\psi| (I_R \otimes M_\ell) X_{RA} \underbrace{(I_R \otimes M_\ell^\dagger)|\psi\rangle_{RB}}_{\equiv |\tilde{\psi}\rangle_{RA}} = {}_{RA}\langle\tilde{\psi}| X_{RA} |\tilde{\psi}\rangle_{RA} \geq 0 \quad \text{for each } \ell$$

- TP? $Tr_B[N_{A\to B}(X_A)] = Tr_B\left[\sum_\ell M_\ell X_A M_\ell^\dagger\right]$

$$= Tr_A\left[\sum_\ell M_\ell^\dagger M_\ell X_A\right]$$

$$= Tr_A[X_A]$$

exercise: check this step (use resolution of I)

($\Leftarrow$) this is the hard direction...

- suppose $N_{A\to B}$ is a linear CPTP map, $\mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$
- useful tool: Choi operator

Def^n Let $\mathcal{H}_R \simeq \mathcal{H}_A$ and $|\Gamma\rangle_{RA} \equiv \sum_{i=0}^{d_A-1} |i\rangle_R |i\rangle_A$ (unnormalized max-ent.)

The Choi operator is $(id_R \otimes N_{A\to B})(|\Gamma\rangle\langle\Gamma|_{RA}) = \sum_{i,j=0}^{d_A-1} |i\rangle\langle j|_R \otimes N_{A\to B}(|i\rangle\langle j|_A)$

some choice of O.N. basis for R, A

- $id_R \otimes N_{A\to B}$ is CPTP $\Rightarrow (id_R \otimes N_{A\to B})(|\Gamma\rangle\langle\Gamma|_{RA})$ is a (non-norm'd) state

$\Rightarrow$ can diagonalize

$\therefore$ let $(id_R \otimes N_{A\to B})(|\Gamma\rangle\langle\Gamma|_{RA}) = \sum_{\ell=0}^{d-1} |\phi_\ell\rangle\langle\phi_\ell|_{RB}$ — (1)

for some $\{|\phi_\ell\rangle_{RB}\}_{\ell=0}^{d-1}$, $d \leq d_R d_B = d_A d_B$ (since $R \simeq A$)

- Choi op. gives us the channel-state (Choi-Jamiołkowski) isomorphism
- any $|\psi\rangle_A = \sum_{i=0}^{d_A-1} \psi_i |i\rangle_A = \sum_i \psi_i ({}_R\langle i|\Gamma\rangle_{RA}) = {}_R\langle\psi^*|\Gamma\rangle_{RA}$

$\therefore \; N_{A\to B}(|\psi\rangle\langle\psi|_A) = N_{A\to B}\left({}_R\langle\psi^*|\Gamma\rangle\langle\Gamma|\psi^*\rangle_R \right)$

$= {}_R\langle\psi^*|\,(id_R \otimes N_{A\to B})(|\Gamma\rangle\langle\Gamma|_{RA})\,|\psi^*\rangle_R$

$= \sum_{l=0}^{d-1} {}_R\langle\psi^*|\,(|\phi_l\rangle\langle\phi_l|_{RB})\,|\psi^*\rangle_R$

- define a linear op. $M_l : \mathcal{H}_A \to \mathcal{H}_B$

$|\psi\rangle_A \longmapsto {}_R\langle\psi^*|\phi_l\rangle_{RB}$

and $\quad {}_A\langle\psi|M_l^\dagger = (M_l|\psi\rangle_A)^\dagger = {}_{RB}\langle\phi_l|\psi^*\rangle_R$

$\Rightarrow N_{A\to B}(|\psi\rangle\langle\psi|_A) = \sum_{l=0}^{d-1} M_l|\psi\rangle\langle\psi|_A M_l^\dagger \qquad \text{— (2)}$

linearity $\Rightarrow \; N_{A\to B}(X_A) = \sum_{l=0}^{d-1} M_l X_A M_l^\dagger \quad \forall \; X_A \in \mathcal{L}(\mathcal{H}_A)$

TP $\Rightarrow \; \mathrm{Tr}_B[N_{A\to B}(|i\rangle\langle j|_A)] = \mathrm{Tr}_A[|i\rangle\langle j|_A] = \delta_{ij}$

but (2): $\mathrm{Tr}_B[N_{A\to B}(|i\rangle\langle j|_A)] = \mathrm{Tr}_B[\sum_{l=0}^{d-1} M_l |i\rangle\langle j|_A M_l^\dagger]$

$= \mathrm{Tr}_A[\sum_{l=0}^{d-1} M_l^\dagger M_l |i\rangle\langle j|_A]$

$= \langle j|\sum_{l=0}^{d-1} M_l^\dagger M_l |i\rangle_A$

only consistent if $\sum_l M_l^\dagger M_l = I_A$, as required. — $\square$

Notes (1, 2) is the channel-state (a.k.a. Choi-Jamiolkowski) isomorphism

- 1 : channel $(N) \Rightarrow$ state
- 2 : state $(\sum|\phi_l\rangle\langle\phi_l|) \Rightarrow$ channel

- We were a bit quick, but $M_l^\dagger$ really is a well-defined map from $B \to A$

${}_A\langle\psi|M^\dagger|\chi\rangle_B = {}_{RB}\langle\phi|\psi^*\rangle_R|\chi\rangle_B$

$= {}_R\langle\psi|\,{}_B\langle\chi^*|\phi^*\rangle_{RB}$

$\equiv {}_R\langle\psi|\,(M^\dagger|\chi\rangle_B)_R \qquad$ since $A \simeq R$, relabel

$\therefore \; M^\dagger : |\chi\rangle_B \to {}_B\langle\chi^*|\phi^*\rangle_{AB}$

Super-important: Monotonicity of Relative Entropy

Thm $D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$

- physical content: a channel can only degrade states; at best, they remain as distinguishable as before.

- Pf $\to$ Wilde Thm. 11.8.1