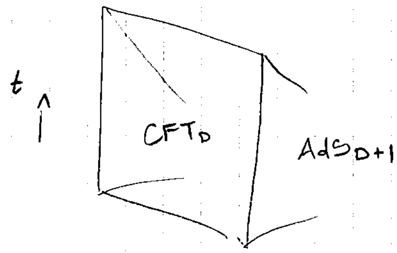
} <u>Talk for Achim's Group</u> Nov 29, 2017

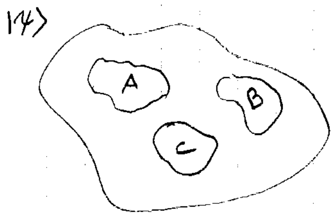
1) <u>Intro</u>: a holographic puzzle

- Recall, AdS/CFT conjecture: certain states of CFT_D dual to a$|$ AdS_{D+1}



- Not all CFT states have a geometric dual, however.
- Natural Q: what fraction of states are holographic?

- One strategy: check holographic entanglement inequalities



let $\rho_A = \mathrm{Tr}_{A^c} |\psi\rangle\langle\psi|$, etc.
$S(A) = -\mathrm{Tr}\, \rho_A \log \rho_A$ etc.

- All states obey, e.g. SSA: $S(AB) + S(BC) \geq S(ABC) + S(B)$

- Not all states obey MMI $S(AB) + S(BC) + S(AC) \geq S(A) + S(B) + S(C) + S(ABC)$
  but holographic states do!

Method 1:
- randomly generate pure state of N qubits (in practice $N=5$)
- trace out k qubits
- check MMI for all perms of A, B, C s.t. $3 \leq |ABC| \leq N-k$

Method 2:
- randomly generate entropy vector $(S_A, S_B, S_C, S_{AB}, S_{BC}, S_{AC}, S_{ABC})$
- check that it actually corresponds to a state
- check MMI

①: almost all states obey MMI
②: $\lesssim 1/2$ of states obey MMI what gives?

A: <u>Ran</u>dom <u>Sta</u>tes are not holographic!

2) <u>Random States: Page's Theorem</u>

Def^n A random state is the random variable $|\psi(U)\rangle = U|\psi_0\rangle$
($|\psi_0\rangle$: ref. state; U: drawn from some prob. distr^n)

Q: Why useful? → parametrize ignorance
→ calculationally useful) easier to avg. over many trials
→ ??

Def^n Let $\mathcal{H}$: $\dim \mathcal{H} = n < \infty$. The normalized Haar measure on $U(n)$ is the unique measure μ s.t.

(1) $\mu(SU) = \mu(US) = \mu(S) \quad \forall \; S \subset U(n), \; U \in U(n)$

(2) $\mu(U(n)) = 1$

$\rightarrow$ μ defines a Haar-random state

Thm (Page) Suppose $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, $\rho_A(U) := \mathrm{Tr}_B |\psi(U)\rangle\langle\psi(U)|$. Then

$$\int d\mu(U) \left\| \rho_A(U) - \frac{I_A}{d_A} \right\|_1 \leq \sqrt{\frac{d_A}{d_B}} \qquad \begin{aligned} d_A &= \dim \mathcal{H}_A \\ d_B &= \dim \mathcal{H}_B \end{aligned}$$

// Aside: Why $\|\cdot\|_1$? $\quad \|\cdot\|_1 : L(\mathcal{H}) \rightarrow \mathbb{R}$

// $\qquad T \mapsto \mathrm{Tr}\sqrt{T^\dagger T}$

// if $\|\rho - \sigma\|_1 < \varepsilon$, then $\|P(\rho - \sigma)\|_1 < \varepsilon \quad \forall$ projectors P $\rightarrow$ measures distinctness

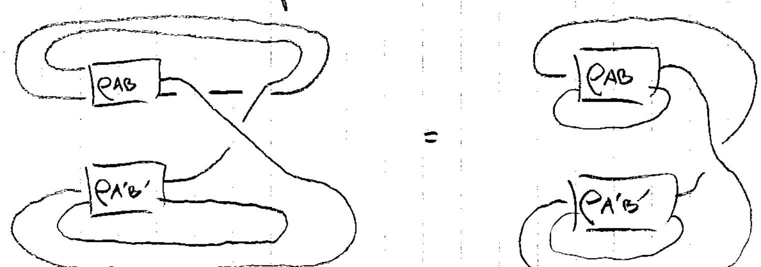
~~A slick proof in 4 steps.~~

① Let $\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|_{AB}$, introduce a copy $A'B'$.

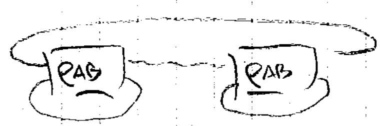
Claim: $\mathrm{Tr}_A \rho_A^2 = \mathrm{Tr}_{ABA'B'} \left[(S_{AA'} \otimes I_{BB'}) |\psi\rangle\langle\psi|_{AB} \otimes |\psi\rangle\langle\psi|_{A'B'} \right]$

SWAP operator

pf. by picture: RHS



LHS



② Let $\langle \cdot \rangle \equiv$ Haar avg. over all pure $|\psi\rangle$

- a result: $\langle |\psi\rangle\langle\psi|_A \otimes |\psi\rangle\langle\psi|_{A'} \rangle = C\, \Pi_{AA'}$

a const $\qquad$ projector onto subspace that is symm under $A \leftrightarrow A'$

// could prove using properties of Haar μ, but should be fairly obvious

// $\rightarrow$ state being avg'd is symm

// $\rightarrow$ avg. cannot depend on any single state

Q: what is C?

$\rightarrow$ $\mathrm{Tr}(C\Pi_{AA'}) = 1 \quad$ so $\quad C = 1/\mathrm{Tr}\,\Pi_{AA'}$

$$\mathrm{Tr}\,\Pi_{AA'} = \sum_{a,a'=1}^{d_A} \langle a|\langle a'|\,\Pi_{AA'}\,|a\rangle|a'\rangle$$

$$= \sum_{aa'} \langle a|\langle a'| \left[\tfrac{1}{2}(I_{AA'} + S_{AA'}) \right] |a\rangle|a'\rangle$$

$$= \tfrac{1}{2}\Big[\sum_{aa'} \langle a|a\rangle\langle a'|a'\rangle + \sum_{aa'} \underbrace{\langle a|a'\rangle\langle a'|a\rangle}_{\delta_{aa'}} \Big]$$

$$= \tfrac{1}{2}\Big[\sum_{aa'} 1 + \sum_{a} 1 \Big]$$

$$= \tfrac{1}{2} d_A (d_A + 1)$$

(3) Now for $|\varphi\rangle_{AB}$,

$$\langle \mathrm{Tr}\,\rho_A^2 \rangle = \langle \mathrm{Tr}_{ABA'B'} [S_{AA'} \otimes I_{BB'}\, |\varphi\rangle\langle\varphi|_{AB} \otimes |\varphi\rangle\langle\varphi|_{A'B'}] \rangle$$

$$= \mathrm{Tr}_{ABA'B'} [S_{AA'} \otimes I_{BB'}\, \langle |\varphi\rangle\langle\varphi|_{AB} \otimes |\varphi\rangle\langle\varphi|_{A'B'} \rangle]$$

$$= \mathrm{Tr}_{ABA'B'} [S_{AA'} I_{BB'} \cdot C \cdot \Pi_{AB,A'B'}]$$

$$= (\ldots)$$

$$= \frac{d_A + d_B}{d_A d_B + 1}$$

(4)
$$\Big\langle \big\| \rho_A - \tfrac{I_A}{d_A} \big\|_2^2 \Big\rangle = \Big\langle \mathrm{Tr}\Big(\rho_A^2 - \tfrac{2}{d_A}\rho_A + \tfrac{I_A}{d_A^2} \Big) \Big\rangle$$

$$= \langle \mathrm{Tr}\,\rho_A^2 \rangle - \frac{2}{d_A} \langle \underbrace{\mathrm{Tr}\,\rho_A}_{1} \rangle + \frac{1}{d_A^2} \langle \overbrace{\mathrm{Tr}\, I_A}^{d_A} \rangle$$

$$= \frac{d_A^2 - 1}{d_A(d_A d_B + 1)} \le \frac{d_A^2}{d_A^2 d_B} = \frac{1}{d_B}$$

- For positive $f(x)$, $\langle \sqrt{f} \rangle \le \sqrt{\langle f \rangle}$

// schematic:
$$\langle \sqrt{f} \rangle = \sum p(x) \sqrt{f(x)}$$
$$= \langle \vec{p}, \vec{\sqrt{f}} \rangle$$
$$\le \|\vec{p}\|_2 \, \|\vec{\sqrt{f}}\|_2 \qquad \text{(Cauchy-Schwarz)}$$
$$= \sqrt{\textstyle\sum (p(x)^2)}\, \sqrt{\textstyle\sum f(x)}$$
$$\le \sqrt{\textstyle\sum f(x)} \le \sqrt{\textstyle\sum p(x) f(x)} = \sqrt{\langle f \rangle}$$

$$\therefore \Big\langle \big\| \rho_A - \tfrac{I_A}{d_A} \big\|_2 \Big\rangle \le \frac{1}{\sqrt{d_B}}$$

- Cauchy-Schwarz again: $\|\cdot\|_1 \le \sqrt{d}\, \|\cdot\|_2$

$$\Rightarrow \Big\langle \big\| \rho_A - \tfrac{I_A}{d_A} \big\|_1 \Big\rangle \le \sqrt{\frac{d_A}{d_B}}$$

3) Back to the puzzle:

The resolution: Haar-typical states generically saturate balanced inequalities

$$S(AB) + S(BC) + S(AC) \qquad S(A) + S(B) + S(C) + S(ABC)$$

$$\overset{\shortparallel}{\log 2 \cdot (\#\text{qubits in } A + \#\text{ qubits in } B)} \qquad \text{etc...}$$

ex: $N = 5$, $k = 1$ A : 1 qubit
B : 1 qubit
C : 2 qubits

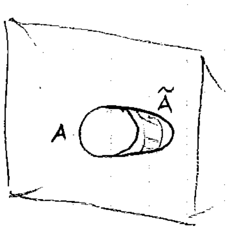
MMI : $\quad 2 + 2 + 2 \; > \; 1 + 1 + 2 + 1$

and importantly, Haar-typical states are not holographic!

Pf: Let $\mathcal{H}$: CFT_D, $|\psi\rangle$ a Haar-typical state, A : subregion w/ characteristic length ℓ (e.g. a D-1 ball)

$\rightarrow$ S(A) is extensive $\qquad S(\rho_A) \propto \left(\frac{\ell}{\varepsilon}\right)^{D-1}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ↳ UV cutoff



but, Ryu-Takayanagi: $\quad S(\rho_A) = \dfrac{\text{area}(\tilde{A})}{4G}$

$\rightarrow$ for suff. small A, $\tilde{A}$ only probes asymp. AdS region

Exercise: For $A \equiv$ D-1 ball in CFT, area of spherical cap $\tilde{A}$ in AdS is

$$\text{area}(\tilde{A}) \lesssim \int_{\varepsilon/\ell}^{1} \frac{d\zeta}{\zeta^{D-1}} \qquad \text{subextensive!}$$

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