Demystifying Wheeler-de Witt

Aidan Chatwin-Davies

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1 Exercise: Minisuperspace

Consider the *minisuperspace* that consists of all closed homogeneous and isotropic FRW spacetimes in $1+3$ dimensions that contain a homogeneous and isotropic real scalar field, Φ. For fun, let's also suppose that there is a (non-negative) cosmological constant Λ that is fixed to some value, *i.e.*, we don't consider is as part of the minisuperspace. In a suitable coordinate system, the line element of the spacetime manifold reads

$$ds^2 = \frac{3}{\Lambda}\left(-dt^2 + a^2(t)d\Omega_3^2\right), \tag{1}$$

where $d\Omega_3^2$ is the round metric on the 3-sphere. In this setting, a 3-geometry and matter configuration is entirely characterized by only two real numbers: the value of the scale factor, b, and the value of the scalar field, Φ_0. As such, the wavefunctional over geometries and matter configurations reduces to a wavefunction with two arguments: $\Psi[\gamma_{ij}(\mathbf{x}), \{\Phi(\mathbf{x})\}] \equiv \Psi(b, \Phi_0)$. In this exercise, we will deduce the form of the Wheeler-De Witt equation that this wavefunction obeys.

(a) In units where $G = \hbar = c = 1$, the Einstein-Hilbert action is

$$S_{EH} = -\frac{1}{16\pi}\int d^4x\sqrt{-g}\,(R - 2\Lambda) \tag{2}$$

and the action of a free scalar field is

$$S_\Phi = \frac{1}{2}\int d^4x\sqrt{-g}\left(g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu} + m^2\Phi^2\right). \tag{3}$$

Use the form of the metric above and the fact that $\Phi(t)$ depends only on t to write down the total action $S = S_{EH} + S_\Phi$. You may find it convenient to express your answer in terms of the following quantities:

$$H^2 \equiv \frac{\Lambda}{3} \qquad \phi \equiv \sqrt{\frac{4\pi}{3}}\Phi \qquad \mu^2 \equiv \frac{3}{\Lambda}m^2 \tag{4}$$

(b) We can read off the Lagrangian $\mathcal{L}$ from the total action $S = \int dt\ \mathcal{L}$. Take a and ϕ to be the canonical coordinates and define their conjugate momenta via

$$p_a = \frac{\partial\mathcal{L}}{\partial\dot{a}} \qquad p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}, \tag{5}$$

where a dot over a quantity denotes a derivative with respect to t. Perform a Legendre transformation and write down the Hamiltonian $\mathcal{H}$ in terms of a, ϕ, p_a, and p_ϕ. Note that $\mathcal{H}$ has no explicit dependence on the coordinate time t; at any given time t_0, $\mathcal{H}$ depends only on the values of $a(t_0) \equiv b$, $\phi(t_0) \equiv \chi$, $p_a(t_0) \equiv p_b$, and $p_\phi(t_0) \equiv p_\chi$. As such, to emphasize that $\mathcal{H}$ is purely a function of 3-geometry and the matter configuration, we can write $\mathcal{H} = \mathcal{H}(b, \chi, p_b, p_\chi)$.

(c) Note that there do not seem to be any secondary constraints for the momenta above, *i.e.*, $\dot{p}_a, \dot{p}_\phi \neq 0$. Why is this? Is it still correct to write $\mathcal{H} = 0$?

(d) Make the identification

$$p_b \to -i\frac{\partial}{\partial b} \qquad p_\chi \to -i\frac{\partial}{\partial \chi} \tag{6}$$

and write down the explicit form of the Wheeler-de Witt equation $\mathcal{H}\Psi(b, \chi) = 0$.

(e) Express the Wheeler-De Witt operator $\mathcal{H}(b, \chi, \partial_b, \partial_\chi)$ in the following form:

$$\mathcal{H} = \frac{1}{2}G^{AB}\partial_A\partial_B + \mathcal{V} \tag{7}$$

G_{AB} and $\mathcal{V}$ are the Wheeler-De Witt metric and potential for minisuperspace in the coordinates $x^A = (b, \chi)$. What is the signature of the Wheeler-De Witt metric?