Demystifying Wheeler - de Witt

Note Title 2016-01-18

Outline:

0. Hamilton - Jacobi Theory
1. H-J Formulation of GR
2. H-J to WdW
3. Example: closed FRW spacetimes

0. Hamilton - Jacobi Theory

Consider, Hamiltonian H, n coords/momenta q_i, p_i

The idea: typically, we consider a Cauchy problem in terms of $q_i(t_0)$, $p_i(t_0)$... HJ theory (at least formally) reformulates the Cauchy problem in terms of boundary values $q_i(t_0)$, $q_i(t_1)$

In practice:

Thm (Jacobi) If $S(q,\alpha,t)$ is any complete sol^n of

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0 \quad \text{— (HJ)}$$

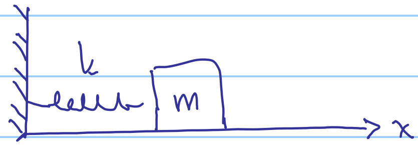
and if $-\beta_i = \dfrac{\partial S}{\partial \alpha_i}$, $p_i = \dfrac{\partial S}{\partial q_i}$ are used to solve for

$q_i(\alpha,\beta,t)$ and $p_i(\alpha,\beta,t)$, then these q and p solve the Hamiltonian problem for $H(q,p,t)$. α_i and β_i are constants that are fcns. of the IC's $q_i(0)$, $p_i(0)$

Note: • the canonical action

$$S(q_0, q_1, t_0, t_1) \equiv \text{ext.} \int_{q_0, t_0}^{q_1, t_1} L \, dt$$

satisfies (HJ), but any solution of (HJ) will give you a sol^n of the original Hamiltonian problem ex Simple Harmonic Oscillator



$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 = h$ a const since conservative

- let $\alpha \equiv h$ then HJ: $\frac{\partial S}{\partial t} + \alpha = 0$

- suggests sol^n $S(x,\alpha,t) = -\alpha t + W(x,\alpha)$

- plug into HJ: $-\alpha + \frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{1}{2}kx^2 = 0$

and $-\beta = \frac{\partial S}{\partial \alpha} \Rightarrow -\beta = -t + \frac{\partial W}{\partial \alpha}$

massage $\Rightarrow$
$$\begin{cases} \frac{\partial W}{\partial x} = \pm\sqrt{2m\alpha - mkx^2} & -(1) \\ t-\beta = \frac{\partial W}{\partial \alpha} & -(2) \end{cases}$$

(1): $W(x,\alpha) = \int_{x_0}^{x} \sqrt{2m\alpha - mk\xi^2}\, d\xi + f(\alpha)$

↳ take this to be zero, changes meaning of β

or $W(x,\alpha) = m\omega \int_{x_0}^{x} \sqrt{a^2 - \xi^2}\, d\xi$ where $\omega^2 = \frac{k}{m}$ $a^2 = \frac{2\alpha}{k}$

Then (2) $\rightarrow$ $t - \beta = m\omega \int_{x_0}^{x} \frac{1}{2}(a^2 - \xi^2)^{-1/2} \cdot \frac{2}{k}$

$$= \frac{1}{\omega}\left[\cos^{-1}\left(\frac{x_0}{a}\right) - \cos^{-1}\left(\frac{x}{a}\right)\right]$$

Q: meaning of β?

consider at $t = t_0$ $\quad t_0 - \beta = 0 \Rightarrow \beta = t_0$

$$\therefore \quad \omega(t - t_0) - \cos^{-1}(x_0/a) = -\cos^{-1}(x/a)$$

or $\quad x = a\cos(\omega(t - t_0) - \phi)$ $\quad$ where $\phi = \cos^{-1}(x_0/a)$

(Also: $\alpha = \frac{p_0^2}{2m} + \frac{1}{2}kx_0^2$, $\quad p(t) = \pm\sqrt{2m\alpha - mkx(t)}$ solve Ham. prob.

1. HJ Formulation of GR

(For more details, see "Quantum Theory of Gravity, I. The Canonical Theory" by De Witt, Phys Rev 160 (5) p. 1113)

Starting point:
$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}$$
$$g^{\mu\nu} = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta^j \\ \alpha^{-2}\beta^i & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{pmatrix}$$

where $\gamma_{ij}\gamma^{jk} = \delta_i{}^k$, $\quad \beta^i = \gamma^{ik}\beta_k$

// c.f. "lapse" $N = -\alpha^2 + \beta_k\beta_k$ and "shift" $N_i = \beta_i$

exercise: show that

$*$ $|i$ is cov. derivative wrt. γ_{ij}

$$\sqrt{-g}\,R = \alpha\gamma^{1/2}(K_{ij}K^{ij} - K^2 + {}^{(3)}R) + (\text{total derivatives})$$

where $K_{ij} = \frac{1}{2}\alpha^{-1}(\beta_{i|j} + \beta_{j|i} - \gamma_{ij,0})$ "extrinsic curvature"

$K^{ij} = \gamma^{ik}\gamma^{jl}K_{kl}$, $\quad K^2 = \gamma^{ij}K_{ij}$

and ${}^{(3)}R \equiv$ Ricci scalar of γ_{ij}, "intrinsic curvature"

i.e. Einstein-Hilbert Action reads

"kinetic" "potential"

$$L = \int \alpha\gamma^{1/2}d^3x\,(K_{ij}K^{ij} - K^2 + {}^{(3)}R)$$

Next, construct Hamiltonian:

- Define $\pi = \dfrac{\delta L}{\delta \alpha_{,0}}$

$\pi^i = \dfrac{\delta L}{\delta \beta_{i,0}}$

$\pi^{ij} = \dfrac{\delta L}{\delta \gamma_{ij,0}}$

Notice! $\pi = 0$, $\pi^i = 0$

"primary constraints"

- Then do the Legendre transformation:

$$H = \int (\pi \alpha_{,0} + \pi^i \beta_{i,0} + \pi^{ij}{}_{,0} \gamma_{ij,0})\, d^3x - L$$

exercise: show that

$$H = \int (\pi \alpha_{,0} + \pi^i \beta_{i,0} + \alpha \mathcal{H} + \beta_i \chi^i)\, d^3x$$

where $\mathcal{H} = \frac{1}{2} \gamma^{-1/2} (\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl}) \pi^{ij} \pi^{kl} - \gamma^{1/2}\, {}^{(3)}R$

$\chi^i = -2\pi^{ij}{}_{|j}$

Notes • $\pi = 0$, $\pi^i = 0$ so we'd may as well drop them from H (they don't affect dynamics)

• α, β_i can be chosen to have definite values / be specified functions of γ_{ij}; amount to coordinate choices

- But notice, $\pi = 0$, $\pi^i = 0$ for all time

$\Rightarrow$ $\pi_{,0} = 0$, $\pi^i{}_{,0} = 0$ "secondary constraints"

- This turns out to be extremely important!

· Consider the Poisson brackets:

$$\begin{aligned}\pi_{,0} &= \{\pi, H\} \\ &= \int \{\pi, \alpha\mathcal{H} + \beta_i \chi^i\} \, d^3x' \\ &= \int \mathcal{H} \underbrace{\{\pi, \alpha\}}_{\delta^3(x-x')} + \chi^i \cancelto{0}{\{\pi, \beta_i\}} \, d^3x' \\ &= \mathcal{H} \overset{!}{\doteq} 0\end{aligned}$$

Similarly $\beta_{j,0} = \{\beta_j, H\} = \delta_{ji}\chi^i \overset{!}{\doteq} 0$

$$\Rightarrow \quad \mathcal{H} = 0 \qquad \text{"Hamiltonian Constraint"}$$
$$\chi^i = 0$$

Transition to Hamilton - Jacobi

· H is a functional of γ_{ij}, π^{ij} i.e.
$$H = H[\gamma_{ij}, \pi^{ij}]$$

· HJ: introduce a HJ functional $S[\gamma_{ij}, \underbrace{\gamma_{ij}^{(0)}}_{\text{"initial condition"}}]$, then the HJ problem becomes to solve

$$0 = H\left[\gamma_{ij}, \frac{\delta S}{\delta \gamma_{ij}}\right], \qquad \pi^{ij} = \frac{\delta S}{\delta \gamma_{ij}}$$

with initial condition $\pi_0^{ij} = \dfrac{\delta S}{\delta \gamma_{ij}^0}$

Note: this formulation is manifestly "time-independent" as it should be in GR! GR only describes events in relation to one another. Here, the "events" being compared are spacelike configurations — given two configs $\gamma_{ij}^{(0)}$ and $\gamma_{ij}^{(1)}$, you can't specify the "time" between them ⟶ GR dynamics "fills in" the spacetime

- Also note that $H=0$ separates into $\mathcal{H}=0$, $\chi^i=0$ which are each individually satisfied.
  - $\mathcal{H}=0$ is all the dynamical content
  - $\chi^i=0$ turns out to be $\Leftrightarrow$ $S[\gamma_{ij}] = S[\tilde{\gamma}_{ij}]$ invariance of S under diffeomorphisms (due to Higgs!)

2. HJ to WdW

Hamilton-Jacobi: $\frac{1}{2}(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl})\pi^{ij}\pi^{kl} - \gamma\, {}^{(3)}R = 0$

Wheeler-de Witt: send $\pi^{ij} \rightarrow -i\dfrac{\delta}{\delta\gamma_{ij}}$

$$\left[\frac{1}{2}(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl})\frac{\delta}{\delta\gamma_{ij}}\frac{\delta}{\delta\gamma_{kl}} + \gamma\, {}^{(3)}R[\gamma_{ij}]\right]\Psi[\gamma_{ij}] = 0$$

This is the (in)famous Wheeler-de Witt equation, $H\Psi = 0$

- Ψ is a wavefunctional on superspace, the space of all spatial 3-geometries (and matter configurations had we included matter)
- Unlike conventional QM, its not clear what amplitude $\Psi[\gamma_{ij}]$ computes.
- Nevertheless, you can use it to compute transition amplitudes

Q: is $H\Psi = 0$ mysterious, no time-dependence?

No!! It was "time"-independent to begin with, as a fully-covariant theory must be!

In fact, it's totally possible to extract time-dependent histories from the HJ / WdW formalism, as must be the case if HJ is really a reformulation of the full Einstein-Hilbert problem.

- Namely, once you have $S[\gamma_{ij}, \gamma_{ij}^{o}]$, histories are given by

$$\pi^{ij} \equiv \frac{\delta L}{\delta \gamma_{ij,0}} \overset{HJ}{=} \frac{\delta S}{\delta \gamma_{ij}}$$

from Lagrangian; explicitly contains x^0

solution of HJ

$\longrightarrow$ ODE for $\gamma_{ij}(x^0, \vec{x})$

Also see:

C. Rovelli; "The Strange Equation of Quantum Gravity"
arXiv: 1506.00927